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DESIGN OF AN ESSENTIALLY NON-OSCILLATORY RECONSTRUCTION PROCEDURE ON FINITE-ELEMENT TYPE MESHES

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ABSTRACT

In this report, we have designed an essentially non-oscillatory reconstruction for functions defined on finite-element type meshes. Two related problems are studied : the interpolation of possibly unsmooth multivariate functions on arbitrary meshes and the reconstruction of a function from its average in the control volumes surrounding the nodes of the mesh. Concerning the first problem, we have studied the behaviour of the highest coefficients of the Lagrange interpolation function which may admit discontinuities of locally regular curves. This enables us to choose the best stencil for the interpolation. The choice of the smallest possible number of stencils is addressed. Concerning the reconstruction problem, because of the very nature of the mesh, the only method that may work is the so called reconstruction via deconvolution method. Unfortunately, it is well suited only for regular meshes as we show, but we also show how to overcome this difficulty. The global method has the expected order of accuracy but is conservative up to a high order quadrature formula only.

Some numerical examples are given which demonstrate the efficiency of the method.

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1 Introduction

During the past few years, a growing interest has emerged for building high order accurate schemes (i.e of order greater that 2) for compressible flows simulations. It is well known that even for smooth initial conditions, these flows may develop discontinuities that make linear schemes useless.

At the beginning of the 80's, the class of Total Variation Diminishing schemes appeared and they have been successfully and widely used with many types of meshes (see for example, [1] for a review and, among many others, [2] for simulations on finite element type meshes). Nevertheless, one of their main weaknesses is that the order of accuracy falls to first order in regions of discontinuity and at extrema, leading to excessive numerical dissipation.

Various methods have been proposed to overcome this difficulty (adaptation of the mesh for example) but one promising way may also be the class of the Essentially Non-Oscillatory schemes (E.N.O. for short) introduced by Harten, Osher and others [3, 4, 5, 6, 7].

The basic idea of E.N.O schemes is to use a Lagrange type interpolation with an adapted stencil : when a discontinuity is detected, the procedure looks for the region around this discontinuity where the function is the smoothest. Then a reconstruction technique may be applied which enables approximation of the function to any desired order of accuracy from its average in control volumes surrounding the mesh points. The approximation is done so that it is conservative.

Some attempts have been made to extend these ideas to multidimensional flows (see for example [8]), but only for structured meshes.

In this report, we intend to study the problem of the reconstruction, up to any order of accuracy, of a given function given either by its value at the nodes of a triangulation or by its averages on control volumes defined around these nodes so that, in the second case, the reconstruction is conservative. This latter problem has already been studied, for smooth functions only, by Barth et al. [9] but their method does not appear to generalize easily to unsmooth functions.

The outline of this report is as follows. In the first part, we give some basic facts about Lagrange interpolation in several dimensions. In particular, we study the problem of the localization of regions of smoothness from the Lagrange interpolation coefficients that generalize those known in one dimension. These results seem to be new. Then, we propose an algorithm for E.N.O. interpolation and we try to give some indications for selection of the smallest family of the possible stencils for second and third order approximation, the only ones considered in this report. We also propose an adaptation of the so-called reconstruction via deconvolution procedure that was originally built for regular meshes, and indicate why, in general, the conservation property must be lost to ensure a high order of approximation. Some numerical tests indicate the performance of this method.

1.1 Notations

- $\mathbb{R}_n[X,Y]$: finite dimensional vector space of two variable polynomials over $\mathbb{R}$,
- $N(n) = \frac{(n+1)(n+2)}{2}$: dimension of $\mathbb{R}_n[X,Y]$,
- $\mathcal{S}^{(n)}$ admissible stencil for solving the Lagrange problem in $\mathbb{R}_n[X,Y]$, see section 2.1.2,
- $||U||$ is the euclidian norm of U,
- $D_{ij}u = \frac{\partial^l u}{\partial^i x \partial^j y}$ where $i + j = l$,
- $D^n u$: n-th derivative of u.

2 Lagrange interpolation on arbitrary sets of $\mathbb{R}^2$

2.1 Sets of admissible points

2.1.1 The polynomials in $\mathbb{R}^2$

We will denote by $\mathbb{R}[X,Y]$ the vector space of the polynomials of two variables (X and Y) with coefficients in $\mathbb{R}$. An element of $\mathbb{R}[X,Y]$ may be described by its (finite) expansion in terms of powers of X and Y :

$$P(X,Y) = \sum_{l=1}^{n} \sum_{i+j=l \ i,j\geq 0} a_{i\,j} X^i Y^j \tag{1}$$

The highest integer such that at least one of the coefficients of the monomials X^iY^j is non zero is called the *total degree* of P.

If (x_0, y_0) is a point of $\mathbb{R}^2$, another expansion of P may be written in terms of the monomials $(X - x_0)^i(Y - y_0)^j$ with the help of the Taylor formula. The total degree of P does not depend on the point (x_0, y_0).

In the sequel, we will denote by $\mathbb{R}_n[X,Y]$ the (finite) vector space of the polynomials of $\mathbb{R}[X,Y]$ with total degree less or equal to n. This vector space has dimension $N(n) = \frac{(n+1)(n+2)}{2}$, a basis of which is the set of monomials $(X - x_0)^i(Y - y_0)^j$ of total degree $i + j$ less or equal to n.

Let us now describe another interesting basis of $\mathbb{R}_n[X,Y]$. Consider (A, B, C) a triangle of $\mathbb{R}^2$ and let us denote by $\Lambda_A, \Lambda_B, \Lambda_C$ the barycentric coordinates of the three points (A, B, C)

defined, for any point M, by :

$$
\begin{aligned}
M &= \Lambda_A \; A + \Lambda_B \; B + \Lambda_C \; C \\
\Lambda_A + \Lambda_B + \Lambda_C &= 1
\end{aligned}
\tag{2}
$$

It is easy to se that, for any pair of points, say A and B, the set $\left\{\Lambda_A^i \; \Lambda_B^j\right\}_{i+j\leq n}$ is also a basis of $\mathbb{R}_n[X,Y]$.

2.1.2 The Lagrange interpolation problem in $\mathbb{R}^2$

The Lagrange interpolation problem may be formulated as follows :

> *Given N and n, two integers, a family of N points in $\mathbb{R}^2$, $\mathcal{S}^{(n)} = (A_i)_{1\leq i\leq N}$ and N real values $(u_i)_{1\leq i\leq N}$, find an element P of $\mathbb{R}_n[X,Y]$ such that for any point A_i of $\mathcal{S}^{(n)}$, one has $P(A_i) = u_i$.*

In the sequel, we will often make no distinction between an element of $\mathcal{S}^{(n)}$, A_l, and its coordinates in a suitable frame, (x_l, y_l).

For this problem to have a solution, two conditions must be fulfilled :

1. one must have $N = \dfrac{(n+1)(n+2)}{2}$
2. the following generalized Van der Monde determinant must be non zero :

$$
\Delta_{\mathcal{S}^{(n)}} = det\left[x_l^i \; y_l^j\right]\Bigg|_{\substack{i+j\leq n \\ (x_l,y_l)\in\mathcal{S}^{(n)}}} = \begin{vmatrix} 1 & x_1 & y_1 & \cdots & x_1^n & x_1^{n-1}y_1 & \cdots x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_N & y_N & \cdots & x_N^n & x_N^{n-1}y_N & \cdots x_N^n \end{vmatrix}
\tag{3}
$$

We will say that the set $\mathcal{S}^{(n)}$ is *admissible* if $\Delta_{\mathcal{S}^{(n)}} \neq 0$. If a set $\mathcal{S}^{(n)}$ is admissible, there exist $\dfrac{(n+1)(n+2)}{2}$ coefficients , $(a_{i\;j})$, such that the solution of the Lagrange problem is :

$$
P = \sum_{l=1}^{n} \sum_{i+j=l \;\; i,j\geq 0} a_{i\;j} X^i Y^j. \tag{4}
$$

The problem of characterizing the admissible sets has been widely studied, see [10] for example and the references therein.

Remarks :

1. The condition (3) has been given for the basis X^iY^j of $\mathbb{R}_n[X,Y]$. A similar and equivalent condition could have been given for the two other bases we have mentioned. The formula (4), provided that the monomials X^iY^j are replaced by the elements of the new basis, is also true.

2. If *card* $\mathcal{S}^{(1)} = 3$, this condition is nothing more than the one which says that the three points must not be aligned.

3. If we were in $\mathbb{R}$, this determinant would be the classical van der Monde determinant.

4. The set of $\frac{(n+1)(n+2)}{2}$-uplets where the condition (3) is not fulfilled is an algebraic curve of $\mathbb{R}^{N(n)}$ and consequently a closed subset of measure zero in $\mathbb{R}^{N(n)}$.

In the next section, we address the question of the practical calculation of the coefficients $a_{i\ j}$.

2.1.3 Determination of the Lagrange expansion

In this section, we use the monomials X^iY^j for expanding polynomials, but any other basis would be suitable and the results are immediately transferable.

The coefficients of the Lagrange expansion are the solution of the linear $N(n) \times N(n)$ system :

$$u_l = \sum_{l=1}^{n} \sum_{i+j=l\ i,j\geq 0} a_{i\ j} x_l^i\ y_l^j \quad \text{for all } (x_l, y_l) \in \mathcal{S}^{(n)}. \tag{5}$$

Since condition (3) is true, the Cramer formula applied to (5) gives the answer.

Several authors have tried to generalize the Newton formula that make the Lagrange interpolation efficient from a numerical point of view, and a very general answer has been given by Muhlbach [11, 12].

In these papers, he addresses the problem of the Lagrange interpolation by a set of functions $(f_i)_{i\in I}$ on a set of points $(A_i)_{i\in I}$. He calls the set (f_i) a *Cebysev-system* if given any function f, for any pair of subsets of I, L and M, having the same (finite) number of elements, there exist real numbers α_i such that :

$$u_i = \sum_{j\in M} \alpha_j f_j(A_i), \quad \text{for all } A_i \in L. \tag{6}$$

For the sake of clarity, we may assume that $L = M = \{1, \cdots N\}$. He uses the notation :

$$\left[\begin{array}{c|c} f_1 \cdots f_k & \\ A_1 \cdots A_k & f \end{array} \right]$$

for denoting the coefficients of f_k in the development (6). Then, in [12], he shows that if one has a Cebysev system (theorem 4.1 pp. 106) :

$$\left[\begin{array}{c|c} f_1 \cdots f_n & \\ A_1 \cdots A_n & f \end{array} \right] = \frac{\left[\begin{array}{c|c} f_1 \cdots f_{n-1} & \\ A_2 \cdots A_n & f \end{array} \right] - \left[\begin{array}{c|c} f_1 \cdots f_{n-1} & \\ A_1 \cdots A_{n-1} & f \end{array} \right]}{\left[\begin{array}{c|c} f_1 \cdots f_{n-1} & \\ A_2 \cdots A_n & f \end{array} \right] - \left[\begin{array}{c|c} f_1 \cdots f_{n-1} & \\ A_1 \cdots A_{n-1} & f \end{array} \right]}. \tag{7}$$

This expression is a direct generalization of the classical Newton formula. Let us make several comments on this formula when applied to our problem :

1. If one adopts the lexicographic ordering $1, X, Y, X^2, XY, Y^2, \cdots, X^n, X^{n-1}Y, \cdots, XY^{n-1}, Y^n$, the previous formula (7) must be applied $n+1$ times to go from a total degree n to a total degre $n+1$. One must also store quite a lot of terms like

$$\left[\begin{array}{c|c} f_1 \cdots f_k & f \\ A_1 \cdots A_k & \end{array}\right]$$

to build the divided difference table. For example, to go from degree one to degree two, one must evaluate and store C_6^3 approximations of gradients by means of approximations on triangles, combine them to obtain approximations on sets of four points (C_6^4 sets), of five points (C_6^5 sets) and of six points (one set). Moreover, to go from approximation on k points to $k+1$ points, $(k+1) \times (k+1)$ determinants must be evaluated.

2. From a numerical point of view, the basis X^iY^j or $(X-a)^i(Y-b)^j$ are not well suited to calculations. This can easily be seen since for any pair (a, b),

$$\left[x_l^i \, y_l^j\right]\Bigg|_{\substack{i+j \leq n \\ (x_l, y_l) \in \mathcal{S}^{(n)}}} = \left[(x_l - a)^i \, (y_l - b)^j\right]\Bigg|_{\substack{i+j \leq n \\ (x_l, y_l) \in \mathcal{S}^{(n)}}} .$$

If (a, b) is any point of $\mathcal{S}^{(n)}$ and if $h = max_{(x_l, y_l) \in \mathcal{S}^{(n)}}(|x_l - a|, |y_l - b|)$, then Hadamard's inequality shows that :

$$|\Delta_{\mathcal{S}^{(n)}}| \leq h^{\kappa(n)},$$

where $\kappa(n) = 1 + \sum_{l=1,n} p \frac{(p+1)(p+2)}{2} = O(n^4)$ so that one reaches very quickly machine zero though the linear system may be well conditioned.

An alternative to this last point is to use local coordinates such as the barycentric ones. In the E.N.O. method we will develop in a further section, for each point, the natural barycentric coordinates are not known *a priori* so that the work has to be repeated at each interpolation call. If this is included in an iterative algorithm, the cost (and the storage) seems to be much too important at least for the cases we have considered in this report. For all these reasons, we have preferred to use classical inversion techniques for linear systems.

2.2 A recurrence formula

In this section, we wish to show another recurrence formula that enables us to obtain the coefficients of the expansion of total degree n from those of the expansion of total degre less than n in only one step. This recurrence formula may be viewed as another version of that given in [12], theorem 3.1, page 400 and will be useful in section 2.3.

Let us begin with some notations. Let $P_1, \cdots, P_{N(n)}$ be a basis of $\mathbb{R}_n[X,Y]$ such that $P_1, \cdots, P_{N(p)}$, $p < n$, is a basis of $\mathbb{R}_p[X,Y]$. The three bases we have considered in section 2.1.2 are of that kind. Let $A_1, \cdots, A_{N(n)}$ be admissible points. We set :

$$R_L = \left(P_1(A_L) \cdots P_{N(n)}(A_L)\right)^T$$

so that the solution of the Lagrange problem (where the u_i's are given),

$$u_i = \sum_{1 \leq j \leq N(n)} \alpha_j P_j(A_i) \quad \text{for all } A_i \in L,$$

may be seen as the solution of the linear system

$$\mathcal{M}\left(\alpha_1 \cdots \alpha_{N(n)}\right)^T = U = \left(u_1 \cdots u_{N(n)}\right)^T, \tag{8}$$

where the Lth row of $\mathcal{M}$ is R_L. The solutions of system (8) are :

$$\alpha_L^n = \frac{det(R_1 \cdots \widehat{R_L} \cdots R_{N(n)})}{det(R_1 \cdots R_L \cdots R_{N(n)})}, \tag{9}$$

where $\widehat{R_L} = U$ denotes the L-th column.

Lemma 2.1 *Let $(A_i)_{1 \leq i \leq N(n)}$ be an admissible set in which any of its $N(p), p \leq n$ subsets is admissible. Then, for a given $p < n$, let $I = \{i_1, \cdots i_{N(p)}\}$ and J be ordered sets such that $I \bigcup J = \{1, \cdots N(n)\}$. If $A = (a_{i\ j})$ is a $N(n) \times N(n)$ matrix, we set*

$$det(A)_I = det(a_{i\ j})\Big|_{\substack{1 \leq i \leq N(p) \\ j \in I}},$$
$$\text{and} \quad det(A)_J = det(a_{i\ j})\Big|_{\substack{N(p)+1 \leq i \leq N(n) \\ j \in J}}.$$

Let us also denote by $\alpha_{L'I}^{(p)}$ the coefficient of $P_{L'}$ of the Lagrange problem of degree p for nodes in I (for degree n, we omit the subscript I).

Then for any $L' \leq N(p)$, we have

$$\begin{aligned} \alpha_L^{(n)} &= \textstyle\sum_{I, card(I) = N(p)} \lambda_I^{L\ L'}\, \alpha_{L'\ I}^{(p)} \\ \lambda_I^{L\ L'} &= (-1)^{\sigma(I)}\, \frac{det(R_1 \cdots R_{L'} \cdots R_{N'})_I\ det(R_{N(p)+1} \cdots R_{L'} \cdots R_N)_J}{det(R_1 \cdots R_N)}, \end{aligned} \tag{10}$$

where $\sigma(I) = 1 + p(p+1)/2 + \sum_{i \in I} i$. In (10), $R_{L'}$ appears a first time at the L'-th row of $det(R_1 \cdots R_{L'} \cdots R_{N'})_I$ and a second time at the $L-N(p)$-th row of $det(R_{N(p)+1} \cdots R_{L'} \cdots R_N)_J$.

Proof : By switching columns L and L' in (9), one gets :

$$\alpha_L^{(n)} = -\frac{det(R_1 \cdots \widehat{R_L} \cdots R_{L'} \cdots R_{N(n)})}{det(R_1 \cdots R_L \cdots R_{N(n)})}.$$

Then, a direct application of the generalized Lagrange formula (see [13], pp. 19-22) to the previous expression gives

$$det(R_1, \cdots \widehat{R_L} \cdots R_{L'} \cdots R_{N(n)}) =$$

$$\textstyle\sum_{I,card(I)=N(p)} (-1)^{\sigma(I)}\ det(R_1 \cdots \widehat{R_L} \cdots R_{N(p)})_I\ det(R_{N(p)+1} \cdots R_{L'} \cdots R_{N(n)})_J$$

Since any $N(p)$ subset is admissible, one has

$$\alpha_{L'\ I}^{(p)} = \frac{det(R_1 \cdots \widehat{R_L} \cdots R_{N(p)})_I}{det(R_1 \cdots R_{L'} \cdots R_{N(n)})_I},$$

and the results follows immediately. •

Lemma 2.2 *With the assumptions of lemma 2.1, then, for $p < n$, if $N(p) \leq L \leq N(n)$ and $L' < N(p)$,*

$$\sum_{I,card(I)=N(p)} \lambda_I^{L\ L'} = 0.$$

Proof : *Apply the Lagrange formula to*

$$det(R_1 \cdots R_{L'} \cdots R_{N(p)} R_{N(p)+1} \cdots R_{L'} R_{N(n)}) = 0,$$

and interpret the coefficients in terms of α's, all equal to 1, and λ's. The lemma 2.1 gives the result •

2.3 Approximation of smooth and unsmooth function by polynomials

The problem of interest in this section is the following : Let u be a real function defined on an open subset Ω of $\mathbb{R}^2$. We assume that u is n times continuously differentiable on Ω except perhaps on a subset of Ω consisting of a finite collection of locally C^1 curves. Let now $\mathcal{T}$ be a mesh. For each point of $\mathcal{T}$, we consider a Lagrange interpolation of u. Is it possible to localize the regions of smoothness of u from the coefficients of the Lagrange interpolation of u ? The answer is yes, at least for second and third order approximations if additional assumptions are made on the mesh. These assumptions guarantee that one can solve the Lagrange problem for any order from 1 to n, and may be seen as a very natural generalization of classical conditions used in the finite elements theory [14].

For functions defined on $\mathbb{R}$, one knows that the divided differences of u satisfy :

- If u is smooth on an interval I containing $x_1, \cdots x_n$, then there exists $\xi \in I$ such that

$$[x_1, x_2, \cdots, x_n|u] = \frac{f^{(n)}(\xi)}{n!},$$

- if $u^{(k)}$ has a jump $[u^{(k)}]$ on I, one has

$$[x_1, x_2, \cdots, x_n|u] = O([u^{(k)}](x_n - x_1)^{k-n}).$$

In this section, we intend to generalize these relations, and in particular, the second one since this problem seems (surprisingly) not to have been studied yet. The proof appears to be technical, and we have not been able to prove it for any total degree. The proof is divided into two parts. In the first part, we study the case of a stencil $\mathcal{S}^{(n)}$ of $N(n)$ points where u admits two values, 0 and 1. We show here that the polynomial of degree n that interpolates u is *exactly* of total degree n. Then, we define a condition on the stencils that appears to be a generalization of the one that says that triangles must not have too small angles to ensure a uniform error bound for classical finite elements [14]. Then, using Lemmas 2.1 and 2.2, we obtain our result. Let us begin with the case of a stencil in which the convex hull u is smooth.

2.3.1 Case of a "smooth" stencil

This problem has been studied by for example Ciarlet and Raviart in [15]. Let us recall one of their main results :

Theorem 2.3 *Let $\Sigma = \{a_i\}_{i=1}^N$ be an admissible (for degree k) set of points of $\mathbb{R}^n$, and let h and ρ be respectively the diameter of Σ and the suprenum of the diameters of the spheres contained in the convex envelop K of Σ. Let u be a function that admits everywhere in K a $k+1$th derivative $D^{k+1}u$ with*

$$M_{k+1} = sup\{||D^{k+1}u(x)||; x \in K\} < +\infty.$$

If P is the unique interpolating polynomial of degree $\leq k$ of u, we have for any integer m with $0 \leq m \leq k$,

$$sup\{||D^m u(x) - D^m P(x)||; x \in K\} \leq C M_{k+1} \frac{h^{k+1}}{\rho^m},$$

for some constants

$$C = C(n, k, m, \Sigma).$$

Moreover, if Σ' is obtained from Σ by an affine transformation, then $C(n, k, m, \Sigma) = C(n, k, m, \Sigma')$.

From this inequality, one sees that the "flatter" Σ is, the poorer the estimation is. A direct application of this theorem gives a generalization of our first statement.

2.3.2 Case of an "unsmooth" stencil

Study of a simplified problem

Let us consider $\mathcal{S}^{(n)}$ an admissible stencil of cardinality $\frac{(n+1)(n+2)}{2}$ and $\mathcal{S}_0, \mathcal{S}_1$ two non-empty subsets of $\mathcal{S}^{(n)}$ having empty intersection, the union of which is $\mathcal{S}^{(n)}$. Let us consider a polynomial P of total degree n such that for all points of $\mathcal{S}_0$, P has value 0 and for those of $\mathcal{S}_1$, P has value 1. Then we conjecture that :

Conjecture 2.1 *If $\mathcal{S}^{(n)}$ is admissible, then the total degree of P is exactly n.*

We have not found in the literature any general proof of the statement, but we have the following lemma :

Lemma 2.4 *If any subset of cardinality $\frac{(k+1)(k+2)}{2}$, $k \leq n$, of $\mathcal{S}^{(n)}$ is admissible, then the conjecture 2.1 is true for $n = 1, 2$*

Proof : The proof will be given for degree 1,2. We also indicate the difficulties for higher degree.

- Degree one. The stencil is made of three points A, B, C that form a triangle. P is either of type Λ_A or $1 - \Lambda_A$ and is of degree exactly one.

- Degree > 1. Let us assume that P is at most of degree $n-1$. Set $N = card(\mathcal{S}_0)$ and $M = \mathcal{S}_1$. We have $N + M = N(n)$. One may assume that $N \leq M$ by changing P into $1 - P$. So, $2N \leq N(n)$. In the following table, we give the maximum values of N up to degree 6 :

degree	$\frac{(n+1)(n+2)}{2}$	N_{max}
2	6	3
3	10	5
4	15	7-8
5	21	10-11
6	28	14

From this table, one can see that there is always, for degree 2, at least 3 points that have the same value. Since these three points are admissible for degree one and since by assumption, P is either a constant or of degree 1, we see that it *must* be a constant which is absurd since it takes two different values. The same argument applied to degrees n=3,4,5 shows that P must be of degree exactly $n-1$ because we always have

$$\frac{(n+1)\,(n+2)}{4} \geq \frac{n\,(n+1)}{2},$$

but fails for degrees greater or equal to 6 (because the previous inequality does not hold if $n > 5$).•

With this in hand, we get the following result, if $R_{i\,j}$ is the following vector :

$$R_{i\,j} = \left((x_1 - x_0)^i (y_1 - y_0)^j \cdots (x_N - x_0)^i (y_N - y_0)^j\right)^T .$$

Lemma 2.5 *Let (x_0, y_0) be any point of the convex hull Σ of $\mathcal{S}^{(n)}$. Assume that conjecture 2.1 is true. Let P be a polynomial that is 1 on $\mathcal{S}_1$ and 0 on $\mathcal{S}_0$ where both sets satisfy the assumptions of lemma 2.4. If $h = max\{(|x_l - x_0|, |y_l - y_0|), (x_l, y_l) \in \mathcal{S}^{(n)}\}$, and if $\mathcal{S}^{(n)}$ satisfies, for some $\alpha > 0$,*

$$Min\left\{x_0 \in \Sigma, |det\left[\frac{R_{0\,0}}{||R_{0\,0}||} \cdots \frac{R_{0\,n}}{||R_{0\,n}||}\right]|\right\} > \alpha, \tag{11}$$

then there exist two constants $C_1(n, \alpha)$ and $C_2(n)$ such that the coefficients of the Taylor expansion of P around (x_0, y_0) :

$$P = \sum_{i+j \leq n} a_{i\,j} (X - x_0)^i (Y - y_0)^j,$$

satisfy

$$C_2(n) h^{-n} \geq \sum_{i+j=n} |a_{i\,j}| \geq C_1(n, \alpha) h^{-n}. \tag{12}$$

Remark :

1. The set of points we define is clearly not empty because, on the convex hull of $\mathcal{S}^{(n)}$ (which is compact), the function defined by the right hand side of (11) is continuous.
2. If $\mathcal{S}^{(n)}$ were a triangle, for example, the minimum of that function would be the minimum of the absolute value of the sines of its angles.

Proof : We adopt the lexicographic ordering of monomials. The set of admissible stencils satisfying condition (11) and $|h| \leq C$ is a compact subset $\mathcal{C}$ of $\mathbb{R}^{N(n)}$. Let us consider the real functions defined on $\mathcal{C}$ by :

$$\phi_{i\,j}(A_1, \cdots, A_N) = \sum_{i=1,n} |det(R_{00} \cdots U_{i\,j} \cdots R_{0n})|,$$

and

$$\psi_{ij}(A_1 \cdots A_N) = \left|\frac{det(R_{00} \cdots V_{ij} \cdots R_{0n})}{det(R_1 \cdots R_N)}\right|,$$

in which the vector U_{ij} stands at the "ij"-th column and has the value

$$U_{ij} = (0, \cdots, 1, \cdots 0)^T .$$

Let also $V_{i\,j}$ be $\sum_{i+j\leq n} P(A_{ij})U_{ij}$ where the "1" is at the L-th position (we refer to the lexicographic order). Let us note that $P(A_{ij})$ is zero or one, and its value depends only on ij and not $\mathcal{S}^{(n)}$.

It is clear that ψ_{ij} is the coefficient of $(X-x_0)^i(Y-y_0)^j$ in the Taylor expansion of P. It is also clear that

$$h^{i+j}\ |det[R_{0\,0}\cdots R_{i\,j}\cdots R_{0\,n}]| \leq \phi_{i\,j},$$

by the triangle inequality. So,

$$h^n\ \theta(A_1,\cdots,A_N) = h^n \sum_{i+j=n} |a_{i\,j}| \geq \sum_{i+j=n} \left|\frac{det(R_{00}\cdots,V_{ij}\cdots R_{0n})}{\phi_{i\,j}}\right|.$$

The left hand side of this inequality is a continuous function on C and hence reaches its minimum. This minimum cannot be zero because that would mean that all of the $a_{i\,j}$'s are zero which is absurd by assumption, and from lemma 2.2.

Now let us turn to the second inequality. We have

$$h^n\ \theta(A_1,\cdots,A_N) = h^n \sum_{i+j=n} |a_{i\,j}| = \sum_{i+j=n} |\psi_{i\,j}|\,.$$

The left hand side is also a continuous function on $\mathcal{C}$ and is bounded above. Clearly, the latter constant does not depend on α. •

Corollar 2.6 *With the assumptions of lemmas 2.4 and 2.5, let n and p be integers satisfying $p<n$. Choose L and L' such that $N(p-1)<L'\leq N(p)$ and $N(n-1)<L\leq N(n)$. Then there exist two constants C_1 and C_2 such that*

$$C_2(n)h^{-n+p} \geq \lambda_I^{L\ L'} \geq C_1(n,\alpha)h^{-n+p},$$

for any subset of cardinality p.

Proof : We apply the definition of $\lambda_I^{L\ L'}$ and the same techniques used in the previous lemma •

Now, we may state our main result :

Theorem 2.7 *Let $\mathcal{S}^{(n)}$ be a stencil satisfying the assumption of Lemmas 2.4 and 2.5 for $n=1,2$. Let u a real function defined on an open subset of Ω in $\mathbb{R}^2$ being C^1 except perhaps on a locally C^1 curve where its n-th order derivative may have a jump $[D^n u]$ such that the intersection $\mathcal{I}$ of that curve and the convex hull of $\mathcal{S}^{(n)}$ is not empty. Then there exits a constant $C(n,\alpha)>0$ such that the coefficients in the Taylor expansion towards a point of $\mathcal{I}$ satisfy :*

$$\sum_{i+j=n} |a_{i\,j}| \geq C(n,\alpha)\ \frac{[D^n u]}{h^n}. \tag{13}$$

Proof : Let us assume that u admits p-th continuous derivatives but its $p+1$-th ones have a jump on a locally C^1 curve $\mathcal{C}$. Let α^p_L be the coefficients of a Lagrange interpolation of degree p on a suitable subset of $\mathcal{S}^{(n)}$. The recurrence formula of Lemma 2.1 gives :

$$\alpha^n_L = \sum_{I, card I = N'} \lambda^{L\ L'}_I$$

Let $\mathcal{S}_+$ and $\mathcal{S}_-$ be the subset of $\mathcal{S}^{(n)}$ defined by

$$a \in \mathcal{S}_+(\text{ resp. } \mathcal{S}_-) \ \text{ iff } \ D_{ij}u(a) \longrightarrow D^+_{ij}(\text{ resp. } D^-_{ij})$$

These two sets exist because of the topological nature of the curve $\mathcal{C}$ (see Figure 1).

Let $\epsilon > 0$ and x_0 be a point of $\mathcal{I}$. There exists $\eta > 0$ such that for $max\{|a_x - x_0|, |a_x - x_0|\} < \eta$, we have the following development of u for points of $\mathcal{S}_+$:

$$u(a) = \sum_{l=0}^{p} \sum_{i+j=l} D_{ij}u(x_0)(a_x - x_0)^i(a_y - y_0)^j + \sum_{i+j=p+1} \left(D^+_{ij} + o(1)\right)(a_x - x_0)^i(a_y - y_0)^j,$$

where a_x and a_y are the x- and y- components of a and D^+_{ij} is the limit of the $i\ j$ partial derivative of u when $x \longrightarrow x_0$ on the right and $|o(1)| \leq \epsilon$. The same type of development is also true for points of $\mathcal{S}_-$; D^+_{ij} is replaced by D^-_{ij}. The properties of the determinants allow us to get for any subset of cardinality $N(p)$ that, by changing the ordering of the points if necessary,

$$\alpha^{(p)}_{L'I} = \frac{\begin{vmatrix} & & D^+_{ij}(a^1_x - x_0)^i(a^1_y - y_0)^j & & \\ & & \vdots & & \\ & & D^+_{ij}(a^m_x - x_0)^i(a^m_y - y_0)^j & & \\ R_{00} & \cdots & D^-_{ij}(a^{m+1}_x - x_0)^i(a^{m+1}_y - y_0)^j & \cdots & R_{0p} \\ & & \vdots & & \\ & & D^-_{ij}(a^{N(p)}_x - x_0)^i(a^{N(p)}_y - y_0)^j & & \end{vmatrix}}{det(R_{00}\cdots R_{0p})} + \frac{\begin{vmatrix} & & o(1)(a^1_x - x_0)^i(a^1_y - y_0)^j & & \\ & & \vdots & & \\ & & o(1)(a^m_x - x_0)^i(a^m_y - y_0)^j & & \\ R_{00} & \cdots & o(1)(a^{m+1}_x - x_0)^i(a^{m+1}_y - y_0)^j & \cdots & R_{0p} \\ & & \vdots & & \\ & & o(1)(a^{N(p)}_x - x_0)^i(a^{N(p)}_y - y_0)^j & & \end{vmatrix}}{det(R_{00}\cdots R_{0p})}, \tag{14}$$

where the index L' stands for (i,j) in the lexicographic order. For the sake of simplicity, let us denote by μ_I and ν_I the coefficients obtained from the first part of the right hand side of equation 14 by replacing D^+_{ij} by 1 and D^-_{ij} by 0 for μ_I and vice versa for ν_I so that

$$\alpha^{(p)}_{L'I} = \left(D^+_{ij} + o(1)\right)\mu_I + \left(D^-_{ij} + o(1)\right)\nu_I.$$

We have to notice that the sum of μ_I and ν_I is one, and that if all the points of I are on the same side of $\mathcal{C}$, then either μ_I or ν_I is zero. Then, using lemma 2.2 and the above remarks, we have, if $[D_{ij}] = D^+_{ij} - D^-_{ij}$:

$$a_L = [D_{ij}] \sum_{card(I)=p} \mu_I \lambda_I^{L\ L'} + \sum_{card(I)=p} o(1)\mu_I \lambda_I^{L\ L'} + \sum_{card(I)=p} o(1)\nu_I \lambda_I^{L\ L'},$$

so that

$$|[D_{ij}]| \left| \sum_{card(I)=p} \mu_I \lambda_I^{L\ L'} \right| \leq |a_L| + \epsilon \sum_{card(I)=p} (|\mu_I| + |\nu_I|)\, |\lambda_I^{L\ L'}|.$$

We can also get a similar inequality by replacing μ_I by ν_I if the points of I are not on the same side of $\mathcal{C}$ so that :

$$\begin{aligned} h^{p-n}\ |[D_{ij}]| \left|\textstyle\sum_{card(I)=p} \lambda_I^{L\ L'}\right| \ \leq\ & h^{p-n} \left\{ |[D_{ij}]| \ \left|\textstyle\sum_{card(I)=p} \mu_I \lambda_I^{L\ L'}\right| \right. \\ +\ & \left. |[D_{ij}]| \ \left|\textstyle\sum_{card(I)=p} \nu_I \lambda_I^{L\ L'}\right| \right\} \\ +\ & 2\ h^{p-n}|a_L| + 2\epsilon h^{p-n} \textstyle\sum_{card(I)=p} (|\mu_I| + |\nu_I|)\, |\lambda_I^{L\ L'}|. \end{aligned} \tag{15}$$

When all the points of I are on the same side of $\mathcal{C}$ the factor 2 is replaced by 1. Because of inequality (11) and from Hadamard's inequality, one has :

$$|\mu_I| + |\nu_I| \leq \frac{\sqrt{\sum_{a\ \mathcal{S}_+} (a^1_x - x_0)^{2i}(a^1_y - y_0)^{2j}} + \sqrt{\sum_{a\ \mathcal{S}_-} (a^1_x - x_0)^{2i}(a^1_y - y_0)^{2j}}}{\alpha ||R_{L'}||}.$$

This latter expression is bounded above by a constant C because L' corresponds to (i,j) in the lexicographic order, without any additional conditions on the geometry. So, the inequalities of (15), with the help of the first inequality of lemma 2.4 become (up to a factor 2 sometimes) :

$$h^{p-n}\ |[D_{ij}]| \left| \sum_{card(I)=p} \lambda_I^{L\ L'} \right| \leq h^{p-n}\ |a_L| + \epsilon C',$$

for a given constant C'. Now, summing up these inequalities for $i + j = n$, with the help of the second inequality of lemma 2.4 one gets our result for ϵ small enough. •

This result enables us to detect the regions of smoothness from those where a jump in one of the derivatives occurs. It is true when conjecture 2.1 is true, and at least for degree 1 and 2.

3 An E.N.O. Reconstruction Technique

In recent papers, Harten and several other authors [3, 4, 5, 6, 7] have tried to derive numerical methods that are able to achieve a higher order of accuracy than classical TVD methods. There are several versions of these techniques, but they can be generally viewed in the following way : starting from some approximation of a real function u (point values or average values in some control volumes), find a pointwise high order approximation u. Two tools are then used :

- an essentially non oscillatory Lagrange type interpolation of a function w,
- this function w may be u itself if one starts from point values or either the primitive function of u or its convolution product with the characteristic function of a copy of the control volume if one can pass from one to another by a translation.

The latter deconvolution technique can only be applied, at least in its standard version, to regular meshes as shown in section 3.2.

In this section, we want to adapt both tools to the situation of unstructured meshes. At least for the second point, the situation seems at first glance very bad : the reconstruction via primitive function cannot be applied in the case of unstructured meshes because the only solution would be to apply it to integrals over domains like $\mathcal{D}_M = [a_1, b_1] \times [a_2, b_2]$, possibly with a suitable transformation of the plane as in [8], which are not in general the union of control volumes.

Now, the reconstruction via deconvolution technique can only be applied to regular meshes (i.e. meshes where the control volumes are translated form one node to another). In this section, we show how to adapt this technique for irregular meshes.

In what follows, $\mathcal{T}$ is a triangulation of Ω, a domain of $\mathbb{R}^2$, u is a function defined on that domain. Around each node i of $\mathcal{T}$, we may define a control volume C_i in many different ways. An example is (see Figure 2) the control volume whose boundary is the segment joining the centroids G of the triangles (i, j, k) having i as a vertex and the middles I_1, I_2 of the segments of those triangles (type I). Another type of control volume is obtained by considering each triangle as the control volume of its centroid. The triangulation we need to describe the ENO algorithm is not $\mathcal{T}$ but another one built from the centroids of the triangles of $\mathcal{T}$. These control volumes always satisfy :

- $\bigcup_{i\in\mathcal{T}} C_i$ is the numerical domain Ω_h, an approximation of the physical one,
- $C_i \bigcap C_j$ is of empty interior when $i \neq j$ (generally speaking, a collection of segments or an empty set).

3.1 Non oscillatory interpolation

Let us consider $n > 0$. In this section, we show how to generalize the $n+1$-th order E.N.O. interpolation technique exposed in [4, 5] to unstructured meshes. As the results of chapter 2 indicate, we must deal with meshes where the stencils we need satisfy the property of lemmas 2.1 and 2.2. This will be the case for most meshes.

Let $\mathcal{T}$ be a triangulation of Ω, a domain in $\mathbb{R}^2$, and u a function defined on that domain. The results we have obtained in chapter 2 can be summarized as follows :

- if $\mathcal{S}^{(n)}$ is an admissible stencil such that u is smooth in its convex hull, then

  $$\sum_{i+j=n} |a_{i\,j}|,$$

  remains finite,

- if in the convex hull of $\mathcal{S}^{(n)}$, u admits continuously differentiable derivatives only up to the order $k < n$, then

  $$\sum_{i+j=n} |a_{i\,j}| = O([u^{(n)}]\; h^{k-n}),$$

  where h is the diameter of $\mathcal{S}^{(n)}$.

Then, as suggested by [4, 5], the E.N.O. algorithm we propose is to consider $\Pi_1(u)$ defined on C_i by the following recursive algorithm :

For a node i,

1. Let $\{T_i\}$ be the set of triangles of $\mathcal{T}$ having i as a vertex. Consider all the linear interpolations where the T_i's are the stencils. Choose the one, T_{min}, where the sum

   $$\sum_{i+j=1} |a_{i\,j}|,$$

   is minimal. We set $\mathcal{S}^{(1)} =$ the nodes of T_{min}.

2. Let $\mathcal{S}^{(n-1)}$ be the stencils defined at the previous step. Consider all the nodes surrounding $\mathcal{S}^{(n-1)}$ in $\mathcal{T}$ and consider all the stencils obtained form $\mathcal{S}^{(n-1)}$ by adding $n+1$ of the nodes surrounding $\mathcal{S}^{(n-1)}$. Choose the stencil minimizing :

   $$\sum_{i+j=n} |a_{i\,j}|.$$

We have intentionally left the second point imprecise because it is obvious that the number of stencils to consider is in general huge. To give an example, if one considers Figure

3 for which $n = 2$, one sees that possible stencils are the vertices of triangles T_{min} and 3 of the ten other triangles. This can be repeated for each of the three edges of T_{min} and leads to a total of $3 \times C_{10}^3 = 360$ possible stencils ! So, one has to define criteria for choosing the "good" and "bad" stencils. These criteria are essentially heuristic and *a priori* ones .

One that seems natural is that when one considers the control volume around each node, the collection of the control volumes of all points of the stencil should be convex. Another one is that the criteria leads to the smallest possible number of stencils, but the stencils must not be confined in a particular angular area of the plane, in order not to favor any direction.

With this in mind, two possible sets of stencils for third order interpolation are:

- the nodes of triangle T_{min} plus, for each of its edges, the three additional nodes of triangles T_1, T_2, T_3. This leads to a maximum of three stencils per triangle,
- or the nodes of triangles of T_{min} plus, for each of its edges, the three additional nodes of triangles of
  - T_1, T_2, T_3,
  - T_1, T_2 and T_4 or T_5,
  - T_1, T_3 and T_6 or T_7,
  - T_1, T_9 or T_{10} and one of the six triangles T_2, T_3, T_4, T_5, T_6, T_7.

The second solution leads to a maximum number of 52 stencils once T_{min} has been found. We have made several tests to evaluate the "performance" of each type of stencil. They are given in section 4.

This particular interpolation is $n + 1$-th order accurate; because u is a polynomial of degree less than n, we have $\Pi_1(u) = u$. This property ensures the $n + 1$-th order accuracy [15], and in particular, we have the estimations of theorem 2.3.

3.2 Deconvolution technique revisited

If $(x)_{i\in \mathbb{N}}$ is a regular mesh of R and u is a real valued function on $\mathbb{R}$, the reconstruction by the deconvolution technique consists of applying the previous algorithm, not to u but to

$$\overline{u}_i(y) = \frac{1}{x_{i+1} - x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} u(x + y - x_i)dx, \tag{16}$$

where, as usual, the mesh size $\Delta x = x_{i+1} - x_i$ is constant and $x_{i+1/2} = x_i + \Delta x/2$. In particular, we see that $\overline{u}_i$ does not depend on i and that $\overline{u}(x_i)$ is the average of u on $[x_{i-1/2}, x_{i+1/2}]$. These values are assumed to be known. Let $\Pi_1(\overline{u})$ be the $m + 1$-th order Lagrange interpolation as decribed in the previous section, with $m \geq n$. Then, the idea is to perform a Taylor expansion of $\overline{u}$ and its successive derivatives around x_i, to truncate them at order $n-k$, to replace the values $\overline{u}, \cdots, \overline{u}^{(n)}$ by $\Pi_1(\overline{u})$ and its n successive derivatives, and to replace the values of u by those of $\Pi_2(u)$, the approximation of u we are looking for :

$$
\begin{aligned}
\Pi_1(\overline{u})(x_i) &= \textstyle\sum_{l=1}^{n} \alpha_i^l \dfrac{\Pi_2(u)^{(l)}(x_i)}{l!}, \\
&\vdots \\
\Pi_1(\overline{u})^{(k)}(x_i) &= \textstyle\sum_{l=1}^{n-k} \alpha_i^l \dfrac{\Pi_2(u)^{(l+k)}(x_i)}{l!}, \\
&\vdots \\
\Pi_1(\overline{u})^{(n)}(x_i) &= \alpha_n^l \dfrac{\Pi_2(u)^{(n)}(x_i)}{n!},
\end{aligned}
\tag{17}
$$

where

$$
\alpha_i^l = \frac{\displaystyle\int_{x_{i-1/2}}^{x_{i+1/2}} (x-x_i)^l \, dx}{\Delta x}.
$$

The linear system is easily invertible because the matrix is upper triangular and its diagonal consists of all "1"s. Furthermore, it is shown in [4], for example, that the average value of $\Pi_2(u)$ over $[x_{i-1/2}, x_{i+1/2}]$ is exactly $\overline{u}_i$. Last, this approximation has the desired order of accuracy when u is smooth because polynomials are left invariant by the construction.

This latter point is the fundamental reason why one achieves the expected order of accuracy . Polynomials are left invariant by the construction because the shape of the control cells does not change from one point to another. If this where not the case, i.e if $\Delta x_{i+1/2} = x_{i+1} - x_i$ were not constant, the formula (16) would indeed depend on the point x_i and this property would be lost. In order to show this, we simply consider $u(x) = x$ and a $m+1$-th order interpolation that has values $\overline{u}_i$ at points x_i. Assuming that $\{x_0, x_1, \cdots\}$ is the stencil selected by the E.N.O. algorithm, we have :

$$
\Pi_1(\overline{u}) = \overline{u}_0 + K(x-x_0) + \cdots \quad \text{where} \quad K = \frac{1}{2} + \frac{\Delta x_{3/2} + \Delta x_{-1/2}}{\Delta x_{1/2}}.
$$

When the mesh is not regular, $K \neq 1$ in general. To obtain $\Pi_2(u) = u$, one must have :

$$
\begin{aligned}
\Pi_1(\overline{u})(y) &= u(y) + \alpha_1 u'(y), \\
\Pi_2(\overline{u})(y) &= K = u'(y).
\end{aligned}
$$

The second equation indicates that one must have $K = 1$ which is, in general, not true.

To overcome this problem, we propose the following technique : apply the ENO search algorithm not to $\overline{u}$ defined by equation 16 but to :

$$
\overline{u}(y) = \frac{1}{area(C_{S^{(n)}})} \int_{C_{S^{(n)}}} u(x+y-x_0) dx, \tag{18}
$$

where $\mathcal{S}^{(n)}$ is any possible stencil around the node x_0 that one has to test and $C_{\mathcal{S}^{(n)}}$ is the union of the control volumes of each node in $\mathcal{S}^{(n)}$:

$$C_{\mathcal{S}^{(n)}} = \bigcup_{j \in \mathcal{S}^{(n)}} C_j. \tag{19}$$

Now, one has to evaluate the integral (18) from the average value of u. This cannot generally be achieved for any function, but is possible for the polynomials of $\mathbb{R}_n[X,Y]$, at least in general. This will be true for the $N(n)$ linear forms over $\mathbb{R}_n[X,Y]$ because

$$< P >_i = \frac{1}{areaC_i} \int_{C_i} P(x)\, dx, \qquad \text{forall} \;\; P \in \mathbb{R}_n[X,Y], \tag{20}$$

are independent. For all the meshes we have considered, these linear forms where always independent, so the problem had a solution. If this is true, then one can find coefficients $\alpha_l(y)$, $1 \leq l \leq N(n)$ so that

$$\frac{1}{area(C_{\mathcal{S}^{(n)}})} \int_{C_{\mathcal{S}^{(n)}}} u(x+y-x_0)\, dy = \sum_{l=1}^{N(n)} \alpha_l(y) < u >_i, \tag{21}$$

when u belongs to $\mathbb{R}_n[X,Y]$. If not, the equation (21) gives a $n+1$-th order quadrature formula. With all this, we get the following theorem whose proof is obvious.

Theorem 3.1 *The algorithm defined by the E.N.O. technique with (18), (19),(20) and (21) leaves invariant the polynomials of* $\mathbb{R}_n[X,Y]$ *and hence gives a* $n+1$*-th order approximation of smooth functions. Moreover, this approximation is conservative up to the quadrature formula 21.*

3.3 Some remarks for the practical calculation of the reconstruction

In section 2.1.3, we have discussed the problem of the practical determination of the coefficients of a Lagrange interpolant because the linear systems to be solved have in general small coefficients. The same problem also arises here if one uses the monomials $(X-x_0)^i(Y-y_0)^j$ to determine the $\alpha_l(y)$ of equation 21. This can easily be seen by using Hadamard's inequality as in section 2.1.3.

For a given node x_0, the E.N.O. technique we propose naturally introduces one triangle having that node as vertex, the triangle T_{min} as in Figure 3 . So, as in section 2.1.3, we will use the barycentric coordinates towards that triangle for practical computations.

4 Numerical examples

We have performed several tests on the second and third order E.N.O. interpolation and E.N.O. reconstruction, but we only report the third order results since they are *a priori* more challenging. In particular, we intend to check numerically that the expected order of accuracy is in fact reached for smooth functions.

The two types of stencils have been tested on the latter case. The use of the second type of possible stencils results in a much more expensive approximation (in general, one must test 52 stencils per triangle versus only 3 in the simplest version) and the results have never been dramatically improved. All the results that are presented bellow have been obtained with the 3 stencil version of the method.

In all these examples, we have assumed that the control volumes are of "type I". The practical calculations of the averages in these control volumes have been performed with a 5-th order quadrature formula [14].

The tests on smooth functions will be performed on :

$$u(x,y) = cos(2\pi(x^2+y^2)).$$

We have displayed in Figure 4 the L^∞ error of the interpolation. The plain curve with squares is obtained with the E.N.O. interpolation, the plain curve with circles is obtained with the E.N.O. reconstruction. The dashed line indicates the slope -3. One can see that the expected order of accuracy is indeed reached. The mesh size h has been measured by choosing the largest segment of the triangulation. All the error estimates have been obtained on irregular meshes as the one presented on Figure 5. These meshes are obtained as random perturbations of regular structured meshes. The set of points that one obtains is triangulated by the Brower algorithm to get a Delaunay triangulation. The main difference between such a mesh and the regular structured one is that the number of triangles each node belongs to is different. We also have done the same tests with regular meshes, and we have not seen any degradation of the convergence.

The locally smooth function we have chosen is obtained by a modification of that used by Harten in [7] for example : if ϕ is any angle, let f_ϕ be :

$$f_\phi(x,y) = \begin{cases} \text{if } r \le -\frac{1}{3}, \; f_\phi(x,y) = -r\sin\left(\frac{\pi}{2}r^2\right), \\ \text{if } r \ge \frac{1}{3}, \; f_\phi(x,y) = 2r - \frac{1}{6}\sin(3\pi r), \\ \text{if } |r| < \frac{1}{3}, \; f_\phi(x,y) = |\sin(2\pi r)|, \end{cases} \quad \text{where } \; r = x + \tan(\phi)y, \tag{22}$$

and let u be :

$$\begin{cases} \text{if } \; x \le \frac{1}{2}\cos\pi y, \;\; u(x,y) = f_{\sqrt{\pi/2}}(x,y), \\ \text{if } \; x > \frac{1}{2}\cos\pi y, \;\; u(x,y) = f_{-\sqrt{\pi/2}}(x,y) + \cos(2\pi y). \end{cases} \tag{23}$$

The function defined by (22)-(23) shows discontinuities in the function itself and its first order derivatives; some of the discontinuities are straight lines (never aligned to the mesh), one is a curved line where the jump changes from one point to another. Last, the behaviour of u is basically one-dimensional on the left of the curve $x = \cos \pi y/2$ and really two-dimensional on the right.

A plot of this function is given in Figure 6. One should obtain straight lines and smooth discontinuity transitions contrary to what is shown in the Figure : this is an effect of the graphic device adapted to P_1 interpolation.

In Figures 7 and 8, we have displayed the node values of the E.N.O. reconstruction for two meshes (1600 nodes and 6400 nodes). To better see the behaviour of both approximation techniques, we also present cross-section on three lines : $Y = 0.75$, $Y = 0$ and $Y = -0.45$. The approximations are obtained from the 1600 nodes mesh (see Figure 5). The latter line goes through one of the triple points (see Figure 6). One can see that the various discontinuities and the smooth regions are well captured by both techniques.

To end this section, we must note that the algorithm for choosing the stencil may lead to some difficulties at the boundaries as can be seen in Figure 6 on the left upper corner : the most left upper triangle of the mesh (Figure 5) does not admit any additional points of the type we consider to make a stencil.

5 Conclusion

In this report, we have developed two methods for the reconstruction of a function admitting discontinuities only on regular planar curves from their node values or from their average in control volumes that surround them. In order to give a firm basis to the Essentially Non-Oscillatory interpolation, we have studied the behaviour of the highest order coefficients of the Lagrange interpolation of smooth functions and unsmooth ones for which the discontinuities lie on regular curves. We have also given an adaptation of the so called "reconstruction via deconvolution" to irregular triangulated meshes.

These techniques have been shown to work quite well on smooth and unsmooth functions. In particular, we have shown in these examples that the minimum number of possible stencils was sufficient for our purpose.

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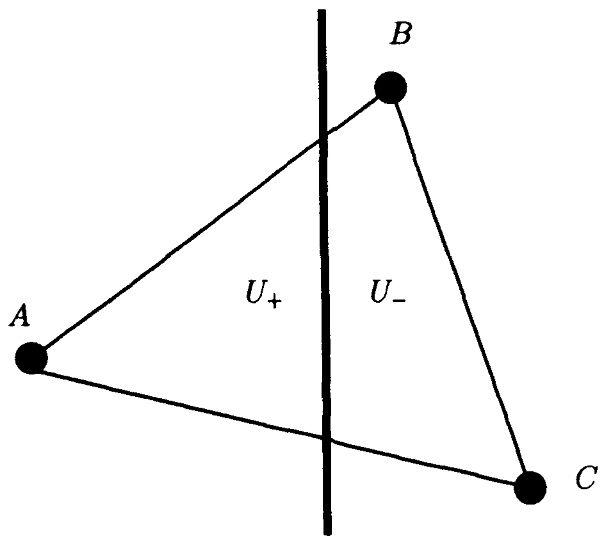


Figure 1: Stencil and discontinuity curve

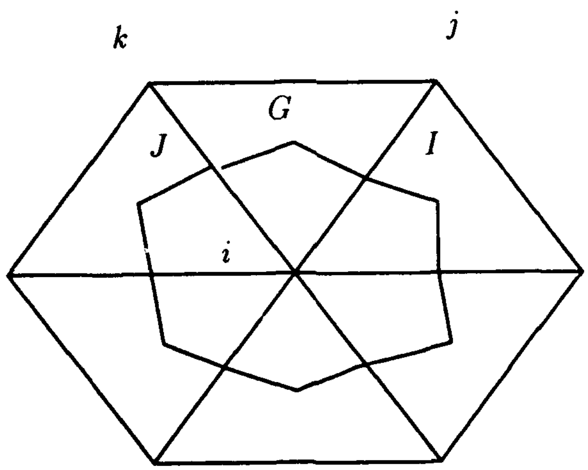


Figure 2: Control volume around i

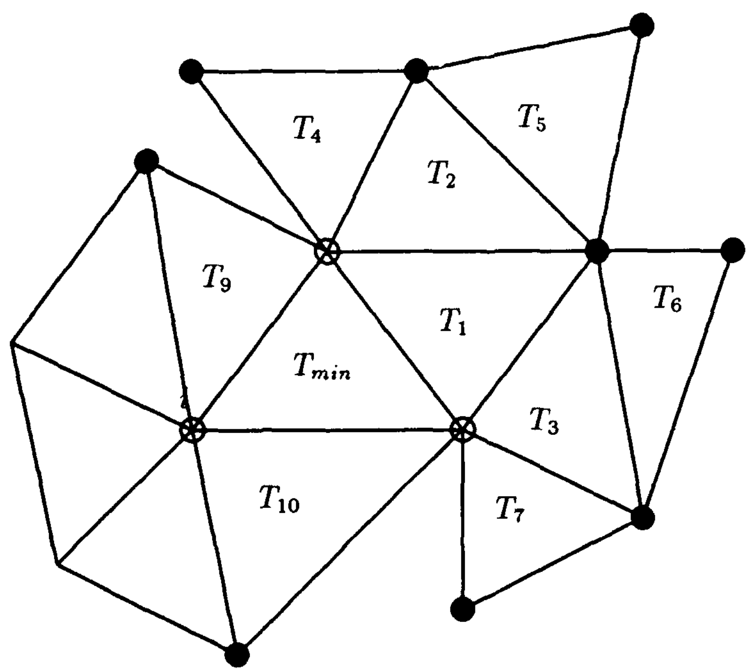


Figure 3: Some possible interpolation points. Circles : points of T_{min} (second order interpolation), black circles : points that may be added to obtain a stencil for third order interpolation.

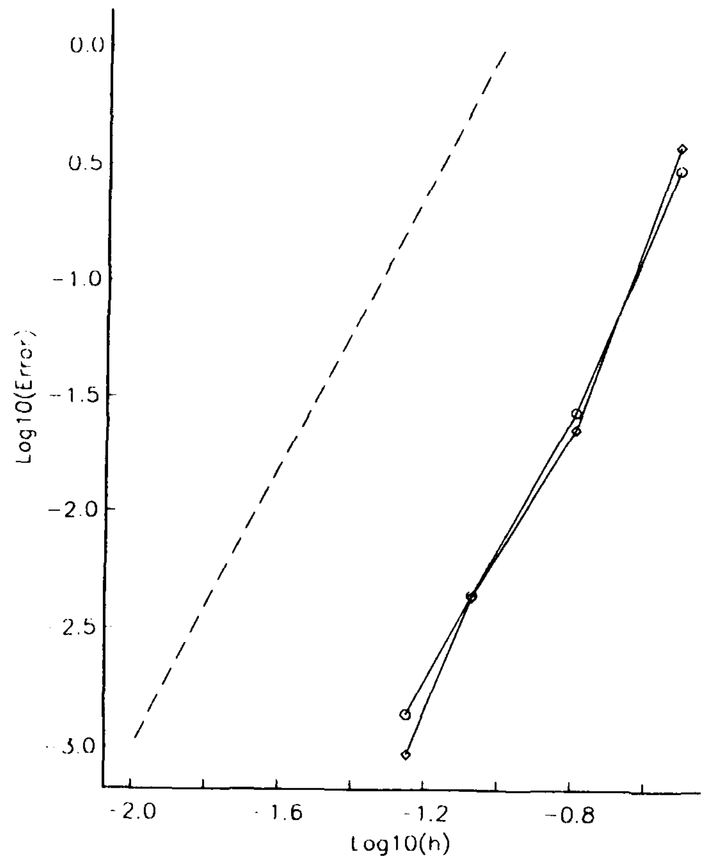


Figure 4: L^∞ error for $f(x,y) = \cos[2\pi(x^2+y^2)]$. Squares : E.N.O. interpolation only, Circles : E.N.O. + reconstruction. Dashed line : slope -3

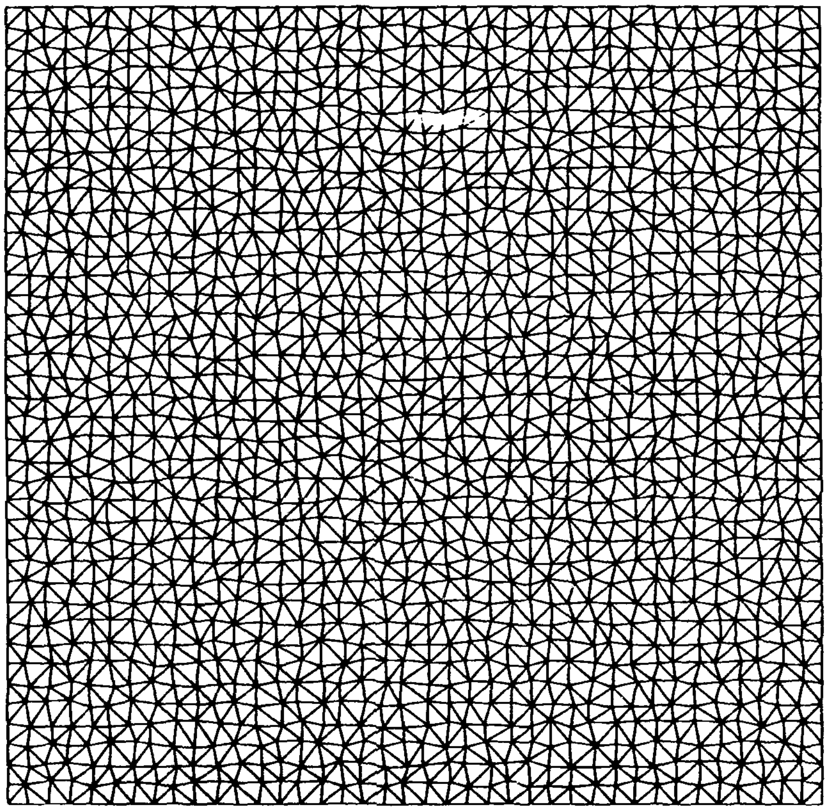


Figure 5: Typical mesh. 1600 nodes, 3042 triangles.

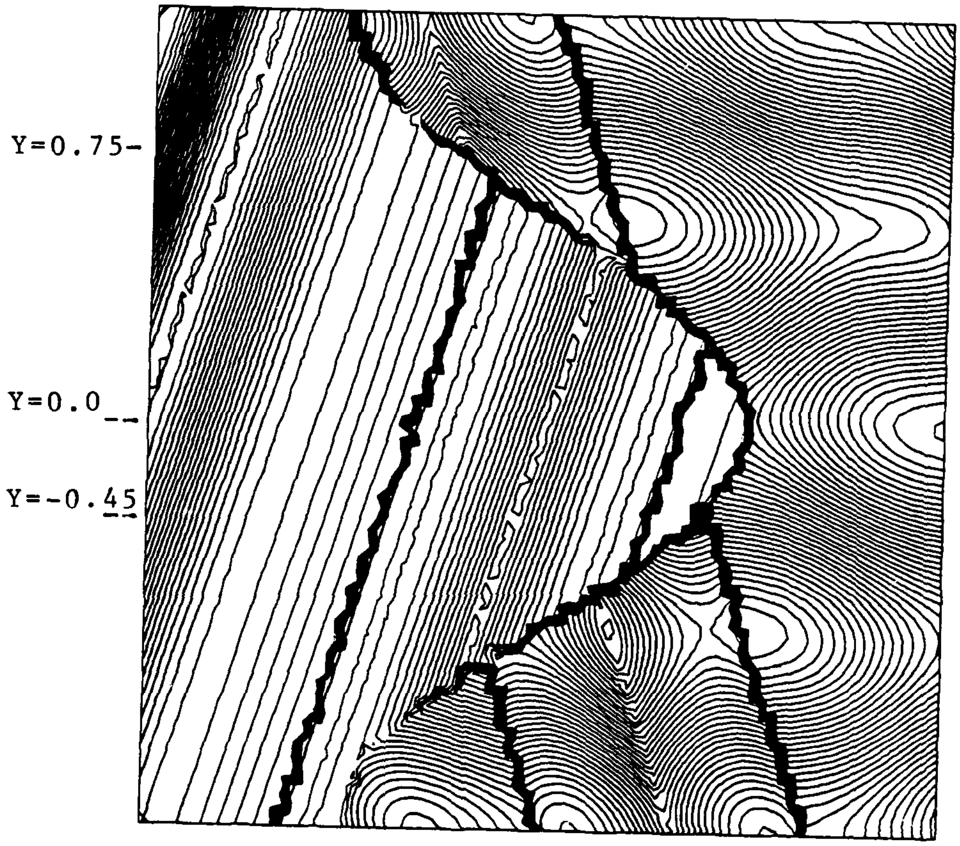


Figure 6: Exact function, Mesh with 6400 nodes. Min=-1.331, Max=2.650, $\delta = 8.292\ 10^{-2}$.

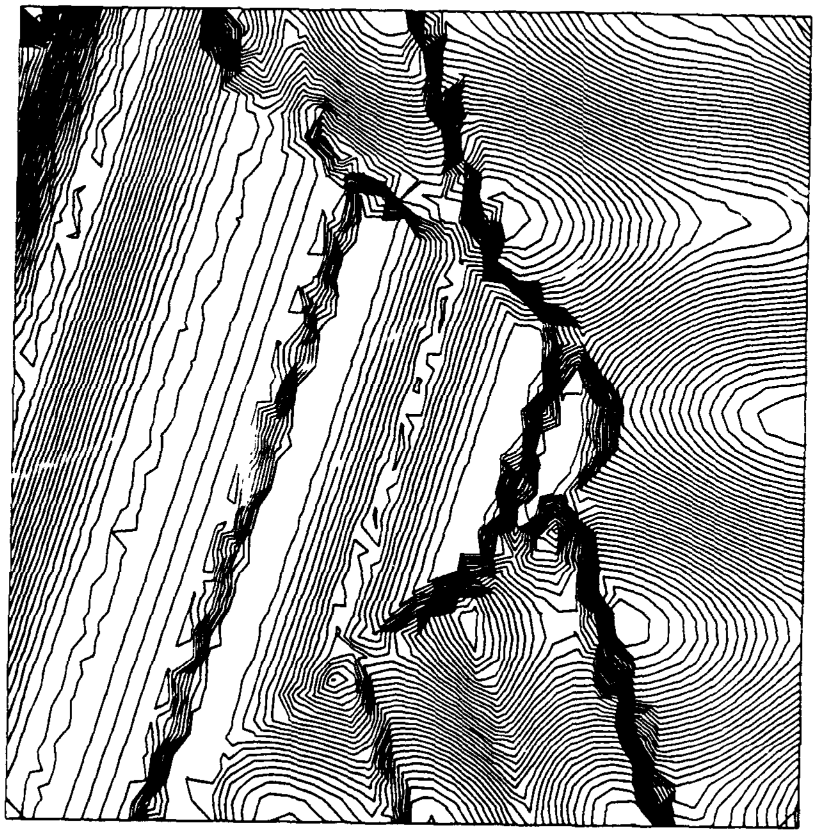


Figure 7: E.N.O. reconstruction with the 1600 nodes mesh. Min=-1.308, Max=2.651, $\delta = 8.249\ 10^{-2}$.

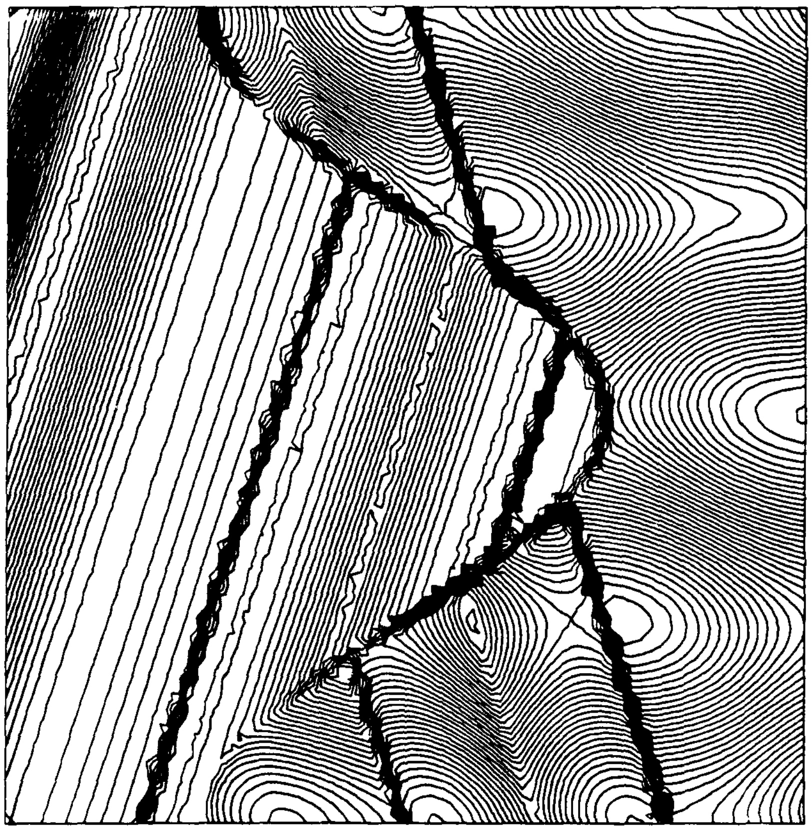
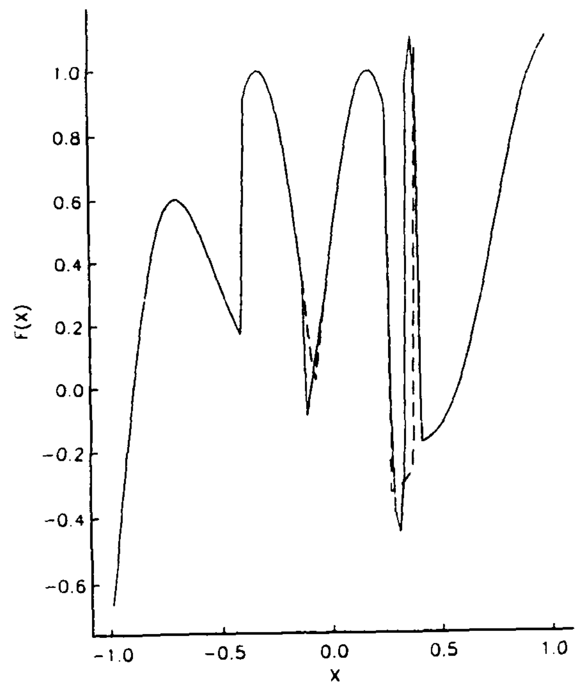


Figure 8: E.N.O. reconstruction with the 6400 nodes mesh. Min=-1.325, Max=2.650, $\delta = 0.8281\ 10^{-2}$.

(a) E.N.O. interpolation.



(b) E.N.O. reconstruction.

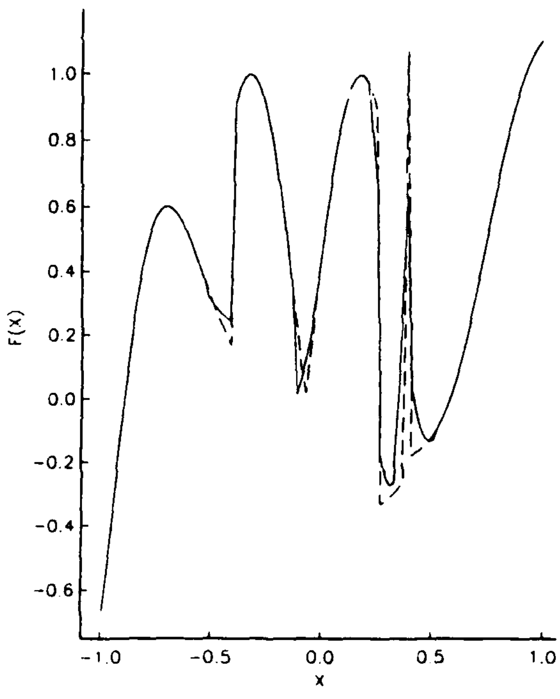
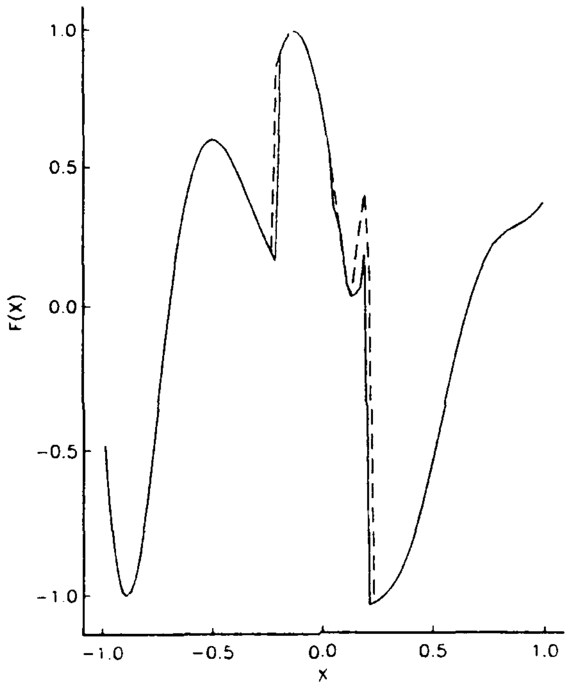


Figure 9: Cross-section at $Y = 0$ of the E.N.O interpolation (a) and the E.N.O. reconstruction (b) for 1600 nodes the mesh.

(a) E.N.O. interpolation.



(b) E.N.O. reconstruction.

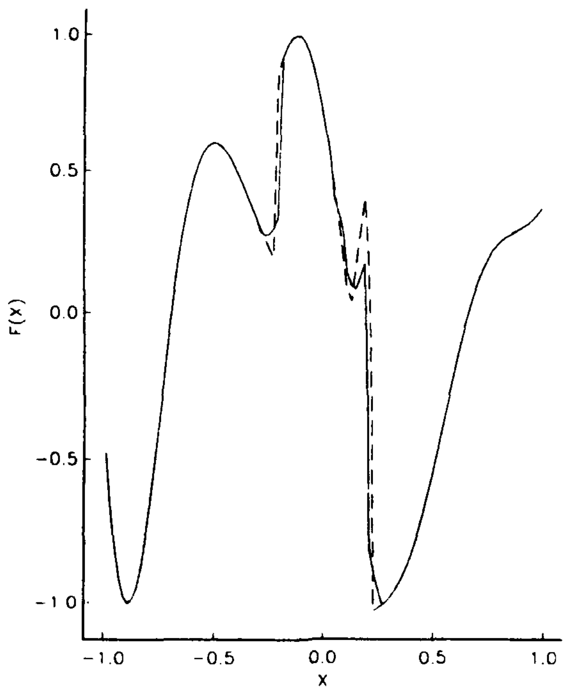
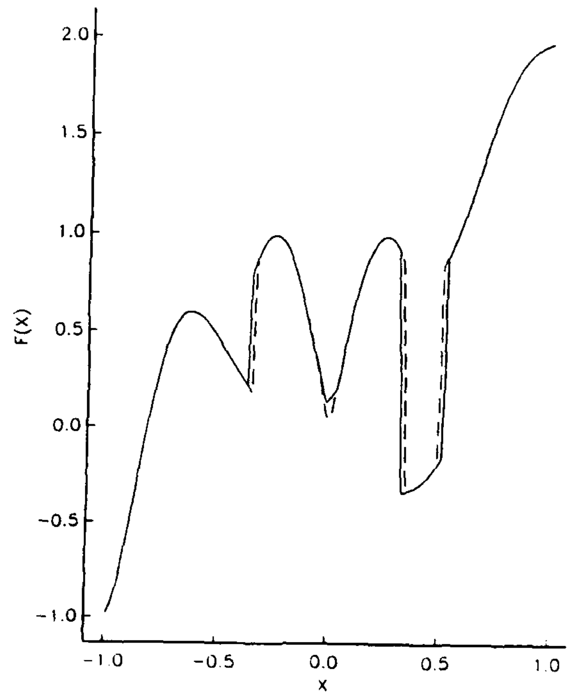


Figure 10: Cross-section at $Y = 0.75$ of the E.N.O interpolation (a) and the E.N.O. reconstruction (b) for the 1600 nodes mesh.

(a) E.N.O. interpolation.



(b) E.N.O. reconstruction.

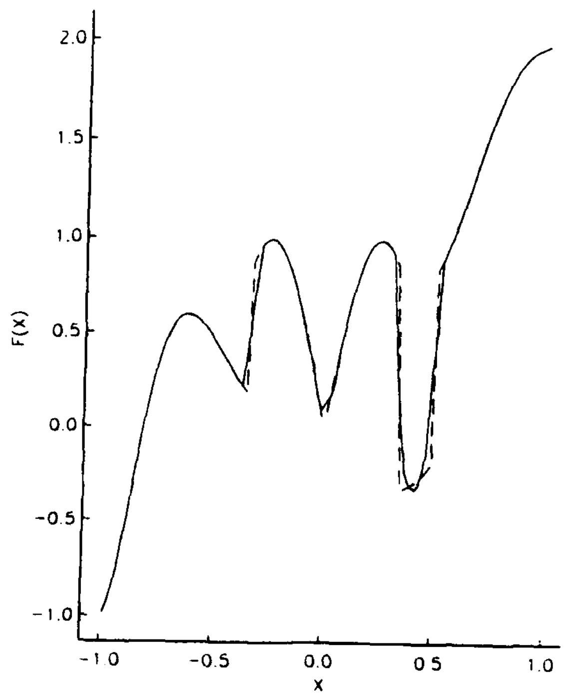


Figure 11: Cross-section at $Y = -0.45$ of the E.N.O interpolation (a) and the E.N.O. reconstruction (b) for the 1600 nodes mesh.

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13. ABSTRACT *(Maximum 200 words)*

In this report, we have designed an essentially non-oscillatory reconstruction for functions defined on finite-element type meshes. Two related problems are studied: the interpolation of possibly unsmooth multivariate functions on arbitrary meshes and the reconstruction of a function from its average in the control volumes surrounding the nodes of the mesh. Concerning the first problem, we have studied the behaviour of the highest coefficients of the Lagrange interpolation function which may admit discontinuities of locally regular curves. This enables us to choose the best stencil for the interpolation. The choice of the smallest possible number of stencils is addressed. Concerning the reconstruction problem, because of the very nature of the mesh, the only method that may work is the so called reconstruction via deconvolution method. Unfortunately, it is well suited only for regular meshes as we show, but we also show how to overcome this difficulty. The global method has the expected order of accuracy but is conservative up to a high order quadrature formula only.

Some numerical examples are given which demonstrate the efficiency of the method.

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